

# Stable Bound Orbits around Black Rings

Takahisa Igata,<sup>\*</sup> Hideki Ishihara,<sup>†</sup> and Yohsuke Takamori<sup>‡</sup>

*Department of Mathematics and Physics, Graduate School of Science,  
Osaka City University, Osaka 558-8585, Japan*

We examine bound orbits of particles around singly rotating black rings. We show that there exist stable bound orbits in toroidal spiral shape near the ‘axis’ of the ring, and also exist stable circular orbits on the ‘axis’ as special cases. The stable bound orbits can have arbitrary large size if the thickness of the ring is less than a critical value.

PACS numbers: 04.50.Gh

## I. INTRODUCTION

Recently, motivated by modern unified theories, gravity in higher dimensions has attracted much interest. In particular, a lot of works are devoted to higher-dimensional black holes (see for a review [1]). The rotating black hole solutions of arbitrary dimensions in vacuum were obtained by Myers and Perry [2], and the singly rotating black ring solutions in five dimensions were derived by Emparan and Reall [3]. It is striking that the black ring solutions reveal that a black hole in vacuum is not specified only by its mass and angular momenta, i.e., higher-dimensional generalization of the black hole uniqueness does not hold in the four-dimensional form. After this discovery, rich varieties of black rings are found by many authors [4].

One of the most important step to study the black holes is the investigation of geodesics in the black hole geometry. It was shown that the geodesic equations are separable for Myers-Perry black holes in arbitrary dimensions [5]. In black ring geometries, geodesics are extensively studied in refs.[6, 7], and it was reported there that the black ring spacetimes hardly admit separability of general geodesics.

It is an interesting question whether a gravitational object has bound orbits of particles. It would

---

<sup>\*</sup>Electronic address: igata@sci.osaka-cu.ac.jp

<sup>†</sup>Electronic address: ishihara@sci.osaka-cu.ac.jp

<sup>‡</sup>Electronic address: takamori@sci.osaka-cu.ac.jp

seem that the existence of stable circular orbits is a characteristic property of the four-dimensional gravity. For simplicity, let us consider the Schwarzschild metric in  $n$  dimensions [8],

$$ds^2 = - \left(1 - \frac{M}{r^{n-3}}\right) dt^2 + \left(1 - \frac{M}{r^{n-3}}\right)^{-1} dr^2 + r^2 d\Omega_{n-2}^2, \quad (1)$$

where  $M$  is the parameter related to the mass of the black hole, and  $d\Omega_{n-2}^2$  is the metric of a unit  $(n-2)$ -dimensional sphere. The effective potential for a free particle in this metric is

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{mM}{2r^{n-3}} - \frac{ML^2}{2mr^{n-1}} + \frac{1}{2}m, \quad (2)$$

where  $m$  and  $L$  are the mass and the total angular momentum of the particle, respectively. In the four-dimensional case,  $n = 4$ , the first term, centrifugal potential, and the second term, gravitational potential, can make a local minimum which corresponds to a stable circular orbit, while there is no stable circular orbit in the case  $n \geq 5$  because the effective potential has no local minimum<sup>1</sup>. The absence of stable circular orbit was also shown in the five-dimensional Myers-Perry black holes [5].

In the present article, we investigate stable bound orbits around the black ring metrics in five dimensions. In the vicinity of thin black rings, the boosted black strings mimic the geometry of the rings. Then, one expects that there exist stable bound orbits near the black ring horizon in spiral shape as in the black string geometry. On the other hand, the gravitational field of the black ring in far region is described by the five-dimensional Schwarzschild metric, approximately. Then, one would expect that there is no stable circular orbit bound by the black rings in this region. Contrary to this intuition, we show that there exist stable bound orbits of which size can be larger than the size of black ring. These are stable toroidal spiral orbits in general, and stable circular orbits as special cases. There exist stable bound orbits of arbitrary large size if the ring thickness is smaller than a critical value.

## II. GEOMETRY OF BLACK RINGS

We consider the metric of black ring in the form

$$ds^2 = - \frac{F(y)}{F(x)} \left( dt - CR \frac{1+y}{F(y)} d\psi \right)^2 + \frac{R^2}{(x-y)^2} F(x) \left[ - \frac{G(y)}{F(y)} d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right], \quad (3)$$

---

<sup>1</sup> In five-dimensional black holes with Kaluza-Klein type [9], there exist stable circular orbits [10]. Even for asymptotically flat case, stable bound states of Nambu-Goto strings are possible [11].

where

$$\begin{aligned} F(\xi) &= 1 + \lambda\xi, \quad G(\xi) = (1 - \xi^2)(1 + \nu\xi), \\ C &= \sqrt{\lambda(\lambda - \nu)\frac{1 + \lambda}{1 - \lambda}}, \end{aligned} \quad (4)$$

and the ranges of the coordinates  $x$  and  $y$  are

$$-\infty \leq y \leq -1, \quad -1 \leq x \leq 1. \quad (5)$$

The parameter  $R$  denotes the radius of the ring, and the parameters  $\nu$  and  $\lambda$  in the range

$$0 < \nu \leq \lambda < 1 \quad (6)$$

describe the thickness of the black ring and rotation velocity in the  $\psi$  direction, respectively. For the regularity at the two axes of rotation,  $\lambda$  has to be chosen as

$$\lambda = \frac{2\nu}{1 + \nu^2}. \quad (7)$$

The event horizon in the topology  $S^2 \times S^1$  is given by  $y = -1/\nu$ .

### III. PARTICLE MOTION IN BLACK RING SPACETIMES

The Hamiltonian of a free particle with mass  $m$  in a metric  $g_{\alpha\beta}$  is generally given by

$$H = \frac{N}{2} \left( g^{\alpha\beta} p_\alpha p_\beta + m^2 \right), \quad (8)$$

where  $N$  is the Lagrange multiplier and  $p_\alpha$  is the canonical momentum. Since  $t$ ,  $\phi$  and  $\psi$  are cyclic coordinates in the case of black ring metric (3), the momenta conjugate to these are constants of motion, say  $p_t = -E$ ,  $p_\phi = L_\phi$  and  $p_\psi = L_\psi$ . Then, we obtain the effective Hamiltonian

$$H = \frac{N}{2} \left[ g^{xx} p_x^2 + g^{yy} p_y^2 + E^2 \left( U_{\text{eff}}(x, y) + \frac{m^2}{E^2} \right) \right], \quad (9)$$

where

$$U_{\text{eff}}(x, y) = g^{tt} + g^{\phi\phi} l_\phi^2 + g^{\psi\psi} l_\psi^2 - 2g^{t\psi} l_\psi \quad (10)$$

with

$$\begin{aligned} g^{tt} &= -\frac{F(x)}{F(y)} - \frac{C^2(x-y)^2(y+1)^2}{G(y)F(x)F(y)}, \quad g^{xx} = \frac{(x-y)^2}{R^2} \frac{G(x)}{F(x)}, \quad g^{yy} = -\frac{(x-y)^2}{R^2} \frac{G(y)}{F(x)}, \\ g^{\phi\phi} &= \frac{(x-y)^2}{R^2 G(x)}, \quad g^{\psi\psi} = -\frac{F(y)(x-y)^2}{R^2 G(y)F(x)}, \quad g^{t\psi} = -\frac{C(x-y)^2(y+1)}{R G(y)F(x)}, \end{aligned} \quad (11)$$

and  $l_\phi := L_\phi/E$ ,  $l_\psi := L_\psi/E$ .

By variation with  $N$ , we get the Hamiltonian constraint condition

$$g^{xx}p_x^2 + g^{yy}p_y^2 + E^2 \left( U_{\text{eff}} + \frac{m^2}{E^2} \right) = 0. \quad (12)$$

We regard the system of a free particle around the black ring as a particle moving in a two-dimensional curved space with the effective potential (10). The system is constrained by the condition (12). For understanding the configuration of particle orbit, it is convenient to use coordinates  $(\zeta, \rho)$  defined by

$$\zeta = R \frac{\sqrt{y^2 - 1}}{x - y}, \quad \rho = R \frac{\sqrt{1 - x^2}}{x - y}, \quad (13)$$

and the effective potential  $U_{\text{eff}}(\zeta, \rho)$  which is given by (10) and (13).

Stationary solutions of the Hamiltonian system (9) are specified by

$$U_{\text{eff}}(\zeta, \rho) + \frac{m^2}{E^2} = 0, \quad (14)$$

$$\partial_\zeta U_{\text{eff}}(\zeta, \rho) = \partial_\rho U_{\text{eff}}(\zeta, \rho) = 0. \quad (15)$$

The position of the stationary solution in the  $\zeta$ - $\rho$  plane is determined by (15) and the particle energy  $E$  for the stationary orbit is given by (14). The stability of the stationary solutions requires that the stationary points are local minimum of  $U_{\text{eff}}(\zeta, \rho)$ .

Typical shapes of the effective potential  $U_{\text{eff}}(\zeta, \rho)$  are shown in Fig.1 as contour plots for the black ring metric with the thickness parameter  $\nu = 0.2$ . There exist local minima of  $U_{\text{eff}}(\zeta, \rho)$  if  $l_\phi$  and  $l_\psi$  are chosen in a suitable finite range<sup>2</sup>. A local minimum point  $(\zeta_s, \rho_s)$ , of which position depends on  $l_\phi$  and  $l_\psi$ , implies a stable bound orbit of particle around the black ring. Each world line is tangent to a timelike Killing vector which is a linear combination of  $\partial_t$ ,  $\partial_\phi$  and  $\partial_\psi$ , and its projection on a timeslice makes a toroidal spiral curve on the two-dimensional torus, direct product of  $S^1$  with radius  $\zeta_s$  and  $S^1$  with radius  $\rho_s$ . Whether the toroidal spiral is closed depends on  $l_\phi$  and  $l_\psi$ . A local minimum point  $(\zeta_s = 0, \rho_s)$  on the  $\rho$ -axis, fixed two-dimensional plane of the rotation generated by  $\partial_\psi$ , corresponds to a stable circular orbit with radius  $\rho_s$ . It is also possible that orbits with  $l_\psi = 0$  are toroidal spirals dragged by rotation of black ring, while orbits with  $l_\psi \neq 0$  are circles.

For a fixed thickness parameter  $\nu$ , the local minima appear in a limited region in the  $\zeta$ - $\rho$  plane. By searching local minimum points numerically, we show that the domain of the stable bound

---

<sup>2</sup> The range of  $l_\phi$  and  $l_\psi$  for stable bound orbits are shown in a separate paper [12].

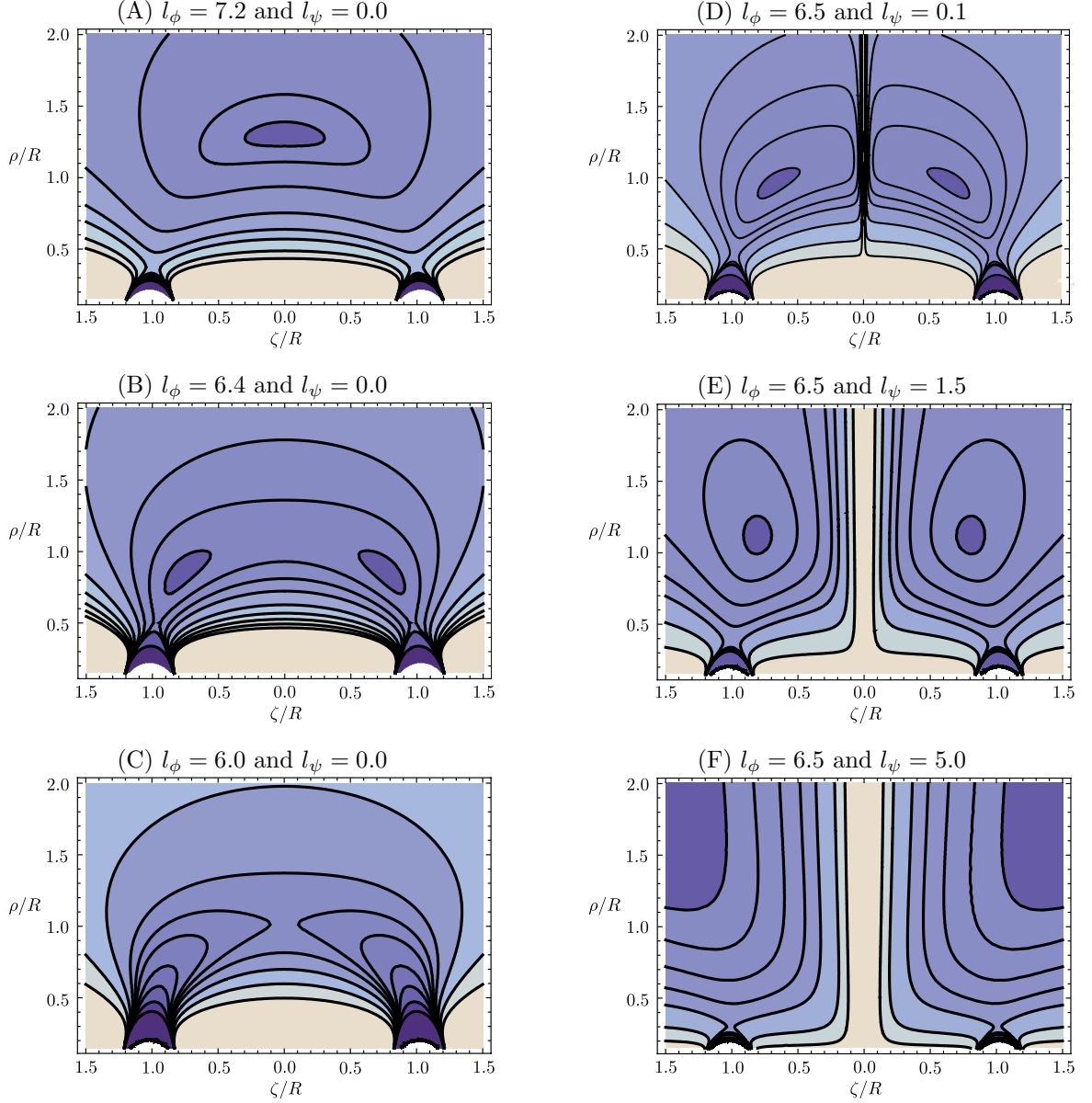


FIG. 1: Contour plots of  $U_{\text{eff}}(\zeta, \rho)$  in the case of the black ring with the thickness parameter  $\nu = 0.2$ . The horizontal axis is  $\zeta/R$  and the vertical axis is  $\rho/R$ . The figures (A),(B),(C) are for  $l_\psi = 0$  cases. In (A) a local minimum exists on the  $\rho$  axis, and in (B) two minima appear off the axis, while in (C) no minimum exists. The figures (D),(E),(F) are for the cases of non-vanishing  $l_\psi$ . Potential barrier near the  $\rho$  axis appears. Two local minima appear in (D) and (E), while no minimum in (F).

orbits for the cases  $\nu = 0.2, 0.4, 0.5$ , and  $0.6$  in Fig.2. There exist stable bound orbits on and near the axis of black ring, and not exists near the equatorial plane. There are two critical values of the thickness parameter  $\nu$ , say  $\nu_\infty$  and  $\nu_0$ . The black rings with the thickness parameter in the range  $0 < \nu < \nu_\infty$  allow stable bound orbits in semi-infinite domains (see Fig.2 for the shape of domain).

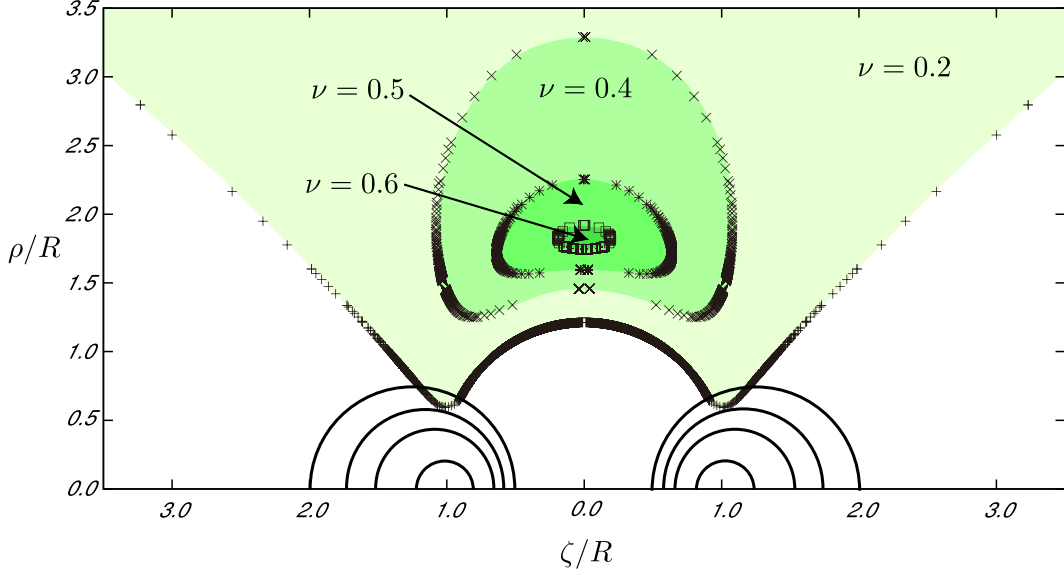


FIG. 2: Domains of stable bound orbits are superposed in the  $\zeta$ - $\rho$  plane as dark regions. The domain, in which potential minima exist, are obtained numerically for the black rings with  $\nu = 0.2, 0.4, 0.5, 0.6$ . Half circles on the horizontal axis denote event horizons of the black rings with  $\nu = 0.2, 0.4, 0.5, 0.6$ , from small semi-circle to large one.

The black rings with  $\nu_\infty < \nu \leq \nu_0$  admit the stable bound orbits in finite domains, and the rings with  $\nu_0 < \nu < 1$  have no stable bound orbit.

The critical values  $\nu_\infty$  and  $\nu_0$  can be obtained if we concentrate on stable circular orbits on the  $\rho$ -axis with  $l_\psi = 0$ . By the regularity at the  $\rho$ -axis, it holds trivially that

$$\partial_\zeta U_{\text{eff}}(\zeta = 0, \rho) = 0, \quad (16)$$

then the stable circular orbits should satisfy the stationary condition

$$\partial_\rho U_{\text{eff}}(\zeta_s = 0, \rho_s) = 0, \quad (17)$$

and the stability conditions

$$\partial_\rho^2 U_{\text{eff}}(\zeta_s = 0, \rho_s) > 0, \quad \partial_\zeta^2 U_{\text{eff}}(\zeta_s = 0, \rho_s) > 0. \quad (18)$$

First, we see the asymptotic form of the effective potential at large distance as

$$U_{\text{eff}}(\zeta = 0, \rho) \simeq -1 - \frac{4R^2\nu - (1-\nu)l_\phi^2}{(1-\nu)^2 \rho^2} + \frac{2\nu R^2 (2R^2 - l_\phi^2)}{(1-\nu)^2 \rho^4}, \quad (19)$$

and

$$\partial_\zeta^2 U_{\text{eff}}(\zeta = 0, \rho) \simeq \frac{4R^2\nu}{(1-\nu)^2 \rho^4} > 0. \quad (20)$$

Then, if two inequalities

$$4R^2\nu - (1 - \nu)l_\phi^2 > 0 \quad \text{and} \quad 2R^2 - l_\phi^2 > 0 \quad (21)$$

hold, all of stationary and stability conditions (17)-(18) are satisfied at

$$\rho_s^2 = \frac{4\nu R^2(2R^2 - l_\phi^2)}{4R^2\nu - (1 - \nu)l_\phi^2}. \quad (22)$$

If  $l_\phi^2$  approaches to  $4R^2\nu/(1 - \nu)$  in the range  $2R^2 > l_\phi^2$ , the radius of the stable circular orbit  $\rho_s$  becomes infinite. It can happen for the cases  $0 < \nu < \nu_\infty := 1/3$ . The positive sign of the  $\rho^{-4}$  term in (19) makes contrast to the negative sign of the  $r^{-4}$  term for  $n = 5$  case in (2). We note that the angular momentum  $L_\phi = l_\phi E$  for the stable circular orbit is finite even if its radius is infinite.

Secondly, we see that the other parameter  $\nu_0$  gives the maximum thickness of the black ring which has the stable bound orbit. By inspecting  $U_{\text{eff}}$ , we find that equations

$$\partial_\rho^2 U_{\text{eff}}(\zeta_s = 0, \rho_s) = 0, \quad \partial_\zeta^2 U_{\text{eff}}(\zeta_s = 0, \rho_s) = 0, \quad (23)$$

hold simultaneously for a value of thickness parameter  $\nu = \nu_0$ . By solving the coupled algebraic equations (17) and (23) for  $\nu = \nu_0, l_\phi = l_0, \rho_s = \rho_0$ , we have<sup>3</sup>

$$\begin{aligned} \nu_0 = & \frac{13}{2} + \frac{1}{2} \left( 145 - 24 \left( \frac{2}{3 + \sqrt{41}} \right)^{1/3} + 6 \left( 4(3 + \sqrt{41}) \right)^{1/3} \right)^{1/2} \\ & - \left[ \frac{145}{2} + 6 \left( \frac{2}{3 + \sqrt{41}} \right)^{1/3} - 3 \left( \frac{3 + \sqrt{41}}{2} \right)^{1/3} \right. \\ & \left. + \frac{1783}{2} \left( 145 - 24 \left( \frac{2}{3 + \sqrt{41}} \right)^{1/3} + 6 \left( 4(3 + \sqrt{41}) \right)^{1/3} \right)^{-1/2} \right]^{1/2} \\ = & 0.65379 \dots, \end{aligned} \quad (24)$$

$$\frac{l_0^2}{R^2} = \frac{3\nu_0^2 - 8\nu_0 + 1}{\nu_0 - 1} + \sqrt{\frac{6\nu_0(\nu_0^2 - 8\nu_0 + 3)}{\nu_0 - 1}}, \quad (25)$$

$$\frac{\rho_0^2}{R^2} = \frac{(1 + \nu_0)^3 - 2(1 - \nu_0)^2 l_0^2/R^2}{(1 - \nu_0)^2 l_0^2/R^2 - 2(1 - \nu_0^2)}. \quad (26)$$

For the black rings with  $0 < \nu < \nu_0$  we can find stationary points which satisfy the stability conditions (18), while the rings with  $\nu_0 < \nu < 1$  the conditions (18) break at all stationary points.

In the parameter range  $0 < \nu < \nu_\infty$ , the potential value of stable circular orbit satisfies

$$-\frac{m^2}{E^2} = U_{\text{eff}}(\zeta_s = 0, \rho_s) < -1 \quad (27)$$

---

<sup>3</sup> The critical value of  $\nu_0$  is obtained in [6], but the meaning is different.

then, the binding energy  $E_b := m - E$  is positive. There appears one more critical value  $\nu_+$  such that if  $\nu_+ < \nu < \nu_0$  the binding energy  $E_b$  of stable circular orbit is negative<sup>4</sup>. This possibility is pointed out in ref.[6]. In the case  $\nu_\infty < \nu < \nu_+$ , the sign of the binding energy depend on the parameter  $l_\phi$ . The critical parameter  $\nu_+$  is obtained algebraically as

$$\begin{aligned} \nu_+ &= \frac{-8(43 - 3\sqrt{177})^{2/3} + (86 - 6\sqrt{177})^{1/3}(-25 + \sqrt{177}) + 2^{2/3}(-68 + 4\sqrt{177})}{-16(43 - 3\sqrt{177})^{2/3} + (86 - 6\sqrt{177})^{1/3}(-41 + \sqrt{177}) + 2^{2/3}(-154 + 10\sqrt{177})} \\ &= 0.52444 \dots \end{aligned} \quad (28)$$

In the limiting case of black ring  $\nu = \nu_0$ , we have the limiting circular orbit with the radius  $\rho_0$  with the angular momentum parameter  $l_0$ . The potential value for the orbit becomes  $-m^2/E^2 = U_{\text{eff}}(\zeta_s = 0, \rho_0) = 0$ , that is, energy  $E$  and angular momentum  $L_\phi = l_0 E$  of the particle diverge. Namely, particles with almost light velocity can be trapped stably in a finite radius by the black ring with the thickness parameter slightly less than  $\nu_0$ .

#### IV. DISCUSSIONS

Black rings in vacuum cannot be distinguished from rotating black holes by their mass and angular momenta in general. However, existence of stable bound orbits, which is not possessed by five-dimensional black holes, is a unique property for black rings with the thickness parameter  $\nu < \nu_0 \approx 0.65379$ . In addition to stable toroidal spiral orbits near black ring horizon, stable circular orbits whose radii are much larger than the ring radius can exist for thin ring with  $\nu < \nu_\infty = 1/3$ . The leading order of gravitational potential at large distance is proportional to inverse square of distance in five dimensions, which is the same order of the centrifugal potential. The next order term, inverse fourth power of distance, is essential for the existence of stable bound orbits. The sign and amount of this term depend on the shape of gravitational source. Thin black rings cause the appearance of stable bound orbits at large distance. Near the stable bound orbits, there exist dynamical orbits bounded in finite regions.

For the black rings with thickness parameter  $\nu < \nu_\infty$ , the binding energy of the bound orbit is positive. Then, a particle falling toward a black ring can be trapped in a stable bound orbit by energy loss as usual. In contrast, for the black rings with  $\nu_+ \approx 0.52444 < \nu < \nu_0$ , all bound orbits have negative binding energies. Though it would be difficult to raise a particle into the bound orbit with negative binding energy, during a violent process of black ring formation highly

---

<sup>4</sup> We check these properties for stable toroidal spiral orbits numerically up to numerical accuracy.



energetic particles would be trapped in stable bound orbits like a storage ring.

The absence of stable bound orbit of test particle around five-dimensional black holes would suggest absence of black hole binary system in five dimensions. In contrast, does the existence of stable bound orbits of particles around black rings suggest the existence of black hole-black ring bound system?

This work is supported by the Grant-in-Aid for Scientific Research No.19540305.

- 
- [1] R. Emparan and H. S. Reall, "Black Holes in Higher Dimensions", Living Rev. Relativity **11**, (2008), 6. URL (cited on 3 Jun. 2010): <http://www.livingreviews.org/lrr-2008-6>.
  - [2] R. C. Myers and M. J. Perry, Annals Phys. **172**, 304 (1986).
  - [3] R. Emparan and H. S. Reall, Phys. Rev. Lett. **88**, 101101 (2002) [arXiv:hep-th/0110260].
  - [4] T. Mishima and H. Iguchi, Phys. Rev. D **73**, 044030 (2006) [arXiv:hep-th/0504018];  
A. A. Pomeransky and R. A. Sen'kov, arXiv:hep-th/0612005;  
H. Elvang and P. Figueras, JHEP **0705**, 050 (2007) [arXiv:hep-th/0701035];  
H. Iguchi and T. Mishima, Phys. Rev. D **75**, 064018 (2007) [Erratum-ibid. D **78**, 069903 (2008)] [arXiv:hep-th/0701043];  
J. Evslin and C. Krishnan, Class. Quant. Grav. **26**, 125018 (2009) [arXiv:0706.1231 [hep-th]];  
K. Izumi, Prog. Theor. Phys. **119**, 757 (2008) [arXiv:0712.0902 [hep-th]];  
H. Elvang and M. J. Rodriguez, JHEP **0804**, 045 (2008) [arXiv:0712.2425 [hep-th]].
  - [5] V. P. Frolov and D. Stojkovic, Phys. Rev. D **68**, 064011 (2003) [arXiv:gr-qc/0301016].
  - [6] J. Hoskisson, Phys. Rev. D **78**, 064039 (2008) [arXiv:0705.0117 [hep-th]].
  - [7] M. Durkee, Class. Quant. Grav. **26**, 085016 (2009) [arXiv:0812.0235 [gr-qc]].
  - [8] F. R. Tangherlini, Nuovo Cim. **27**, 636 (1963).
  - [9] P. Dobiasch and D. Maison, Gen. Rel. Grav. **14**, 231 (1982);  
G. W. Gibbons and D. L. Wiltshire, Annals Phys. **167**, 201 (1986) [Erratum-ibid. **176**, 393 (1987)];  
H. Ishihara and K. Matsuno, Prog. Theor. Phys. **116**, 417 (2006) [arXiv:hep-th/0510094].
  - [10] K. Matsuno and H. Ishihara, Phys. Rev. D **80**, 104037 (2009) [arXiv:0909.0134 [hep-th]].
  - [11] T. Igata and H. Ishihara, Phys. Rev. D **81**, 044024 (2010) [arXiv:0911.5549 [hep-th]];  
T. Igata and H. Ishihara, arXiv:0911.0266 [hep-th].
  - [12] T. Igata, H. Ishihara, and Y. Takamori, in preparation.